Geometric Sequences

- **Geometric sequences** contain a pattern where a fixed amount is multiplied from one term to the next (common ratio \( r \)) after the first term.

- **Geometric sequence examples:**
  - 2, 4, 16, 32, ...
  - Domain: ____________________
  - Range: ____________________
  - Graph shown at right
  - common ratio \( r = ________ \)
  - The graph of a geometric sequence is __________
  - Find the common ratio (\( r \)) for the following geometric sequences:
    - a) 5, 10, 20, 40, ... \( r = ________ \)
    - b) -11, 22, -44, 88, ... \( r = ________ \)
    - c) \( \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \frac{64}{81}, ... \) \( r = ________ \)

- **Identifying Geometric Sequences**
  - Identify whether the following sequences are arithmetic, geometric, or neither. If it is arithmetic, find \( d \) and if it is geometric, find \( r \).
    - a) 4, 10, 18, 28, 40, ... ________________
    - b) 625, 125, 25, 5, 1, ... ________________
    - c) 81, 27, 9, 3, 1, ... ________________
    - d) 1, 2, 6, 24, 120, ... ________________
    - e) -4, 8, -16, 32, -64, ... ________________
    - f) 8, 1, -6, -13, -20, ... ________________
• **Finding Terms in a Geometric Sequence**

  o Find the 7th term in the sequence: 2, 6, 18, 54, ...
    - \( r = \) ____________
    - \( a_7 = \) ____________
  
  o Is there a pattern?
    - \( a_1 = 2 \)
    - \( a_2 = a_1 \cdot r \)
    - \( a_3 = a_2 \cdot r = a_1 \cdot r \cdot r = \) ____________
    - \( a_4 = a_3 \cdot r = a_2 \cdot r \cdot r = a_1 \cdot r \cdot r \cdot r = \) ____________
    - \( a_n = \) ____________

  To find the \( n^{\text{th}} \) term in a geometric sequence:
  \[
  a_n = a_1 \cdot r^{n-1}
  \]
  where \( a_1 \) is the first term of the sequence, \( r \) is the common ratio, \( n \) is the number of the term to find

  o You try...

  a) Find the common ratio \( r \):
  6, -3, \( \frac{3}{2} \), \( -\frac{3}{4} \), ...

  b) Find the common ratio for the sequence given by the formula:
  \( a_n = 5(3)^{n-1} \)

  c) Find the 7th term of the sequence:
  2, 6, 18, 54, ...

  d) Find \( a_8 \) for the sequence:
  0.5, 3.5, 24.5, 171.5, ...

  e) Write a rule for the \( n^{\text{th}} \) term of the sequence, then find \( a_7 \):
  4, 20, 100, 500, ...

  f) One term of a geometric series is \( a_4 = 12 \). The common ratio \( r = 2 \).
  Write a rule for the \( n^{\text{th}} \) term.

  g) Two terms in a geometric sequence are \( a_3 = -48 \) and \( a_6 = 3072 \).
  Find a rule for the \( n^{\text{th}} \) term.
**Geometric Series**

- A geometric series is the sum of the terms in a geometric sequence: $S_n = \sum_{i=1}^{n} a_i r^{i-1}$

**Sums of a Finite Geometric Series**

- The sum of the first $n$ terms of a geometric series is given by:

\[
S_n = a_1 \left( \frac{1-r^n}{1-r} \right)
\]

where $a_1$ is the first term in the sequence, $r$ is the common ratio, and $n$ is the number of terms to sum.

- Why?
  - Expand $S_n = \ldots$
  - Multiply both sides by $r$: \ldots
  - Subtract: \ldots
  - Solve for $S_n$: \ldots

- Examples:
  - a) Find the sum: $\sum_{i=1}^{16} 4(3)^{i-1}$
  - b) Find the sum of the first 8 terms of the sequence: -5, 15, -45, 135, ...
  - c) Find the sum: $\sum_{k=1}^{5} 3^k$

- You Try...
  - a) Find the sum: $\sum_{k=1}^{8} 6(-2)^{k-1}$
  - b) Find the sum of the first 8 terms of the sequence: 6, 24, 96, ...
  - c) A soccer tournament has 64 participating teams. In the first round, 32 games are played. In each successive round, the number of games decreases by one half. Find a rule for the number of games played in the nth round, and the total number of games played.
- **Sums of Infinite Geometric Series**

- Consider the series: \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \)

- Is it a geometric series? _____ What is \( r \)? _____

- Find the first 5 partial sums, \( S_1 \), \( S_2 \), \( S_3 \), \( S_4 \), and \( S_5 \):
  - \( S_1 = \frac{1}{2} = 0.5 \)
  - \( S_2 = \frac{1}{2} + \frac{1}{4} = 0.75 \)
  - \( S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = _____ \)
  - \( S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = _____ \)
  - \( S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = _____ \)

- Graph these partial sums:

- What do you think will happen as we increase \( n \)? ________________________________

- Examine the formula for the partial sum: \( S_n = a_1 \left( \frac{1-r^n}{1-r} \right) \)
  - What happens as \( n \) gets very big (approaches infinity)? Consider values of \( r \):
    - \( r > 1 \) ________________
    - \( r < -1 \) ________________
    - \(-1 < r < 1 \) ________________

- An infinite geometric series will **converge** if \( |r| < 1 \); otherwise it will **diverge**

- Sum of an Infinite Geometric Series Formula
  \[
  S_\infty = \frac{a_1}{1-r}, \text{ when } |r| < 1
  \]
Examples: Find the sum, if possible...

\[ \sum_{i=1}^{\infty} 5(0.8)^{i-1} \]

b) \[ 1-\frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \ldots \]

You Try...Find the sum of the infinite series, if possible...

\[ \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{k-1} \]

b) \[ \sum_{j=1}^{\infty} \left( \frac{5}{4} \right)^{j-1} \]

c) \[ \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots \]

Recursive Formulas

So far, we have worked with explicit formulas for arithmetic and geometric sequences

- The explicit rule for the \( n \)th term of an arithmetic sequence: _________________
- The explicit rule for the \( n \)th term of an geometric sequence: _________________

We can also define terms of a sequence recursively

- Recursive formulas define one or more initial terms, and then each further term is defined as a function of preceding terms.

Examples of recursion:

- Fibonacci sequence
  - Initial terms: \( a_1 = 0, \ a_2 = 1 \)
  - Recursive equation: \( a_n = a_{n-1} + a_{n-2} \)
  - Expand: __________________________________________________________________________

- Factorial function
  - Initial terms: \( 0! = 1 \)
  - Recursive equation: \( n! = n \cdot (n-1)! \) (for \( n > 0 \))
  - Expand: __________________________________________________________________________
• **Recursive formulas for arithmetic and geometric sequences**
  
  o Recursive formula for **arithmetic** sequences: ________________
    ı Add the common difference to the previous term
  
  o Recursive formula for **geometric** sequences: ________________
    ı Multiply the common ratio to the previous term
  
  o Examples: Write a recursive rule for the sequence...
    a) 3, 13, 23, 33, 43, ...
    b) 16, 40, 100, 250, 625, ...
    c) Write the first 5 terms of the sequence:
      \( a_1=3; \ a_n=a_{n-1} - 7 \)
  
  o You Try...
    a) Write the first 6 terms of the sequence:
      \( a_0=1, \ a_n=a_{n-1} + 4 \)
    b) Write the first 6 terms of the sequence:
      \( a_1=1, \ a_n=3a_{n-1} \)
    c) Write a recursive rule for the sequence:
      2, 14, 98, 686, 4802, ...
    d) Write a recursive rule for the sequence:
      19, 13, 7, 1, -5, ...
    e) Write a recursive rule for the sequence:
      1, 1, 2, 3, 5, ...
    f) Write a recursive rule for the sequence:
      1, 1, 2, 6, 24, ...
Formula Summary:

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<tr>
<th>Sequences</th>
<th>Explicit</th>
<th>Recursive</th>
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<tbody>
<tr>
<td>Arithmetic</td>
<td>$a_n = a_1 + (n - 1)d$</td>
<td>$a_n = a_{n-1} + d$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$a_n = a_1 \cdot r^{n-1}$</td>
<td>$a_n = a_{n-1} \cdot r$</td>
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<table>
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<tr>
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<tr>
<td>Sum of first n integers:</td>
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<tr>
<td>Sum of first $n^2$ integers:</td>
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<tr>
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<td>Sum of geometric series:</td>
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<td>Sum of infinite geometric series</td>
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