1. 1985 BC 33

If \( \frac{dy}{dt} = -2y \) and if \( y = 1 \) when \( t = 0 \), what is the value of \( t \) for which \( y = 1/2 \)?

(A) \( -\frac{\ln 2}{2} \)  
(B) \( \frac{1}{4} \)  
(C) \( \frac{\ln 2}{2} \)  
(D) \( \frac{\sqrt{2}}{2} \)  
(E) \( \ln 2 \)

\[
\int \frac{dy}{y} = \int -2 \, dt \\
\ln |y| = -2t + C \\
\frac{1}{2} = e^{-2t} \\
\ln (\frac{1}{2}) = \ln e^{-2t} \\
1 = Ae^0 \\
1 = A
\]

\[
y = e^{-2t} \\
\frac{1}{2} = e^{-2t} \\
\ln (\frac{1}{2}) = \ln e^{-2t} \\
ln (\frac{1}{2}) = -2t \\
t = -\frac{1}{2} \ln (\frac{1}{2}) \\
= -\frac{1}{2}(-\ln 2) \\
= \frac{\ln 2}{2} \\
Choice (C)
\]
2. 1985 BC 44

At each point on a certain curve, the slope of the curve is $3x^2y$. If the curve contains the point $(0,8)$, then its equation is

(A) $y = 8e^{x^3}$
(B) $y = x^3 + 8$
(C) $y = e^{x^3} + 7$
(D) $y = \ln(x+1) + 8$
(E) $y^2 = x^3 + 8$

\[
\frac{dy}{dx} = 3x^2y
\]
\[
\int \frac{dy}{y} = \int 3x^2 \, dx
\]
\[
\ln|y| = x^3 + C
\]

The point $(0, 8)$ is on the curve $e^{\ln y} = e^{x^3 + \ln 8}$

\[
\ln 8 = C
\]
\[
\ln|y| = x^3 + \ln 8
\]
\[
|y| = e^{x^3} \cdot e^{\ln 8}
\]
\[
y = 8e^{x^3}
\]

Choice (A)
3. 1988 BC 39

If \( \frac{dy}{dx} = y \cdot \sec^2 x \) and \( y = 5 \) when \( x = 0 \), then \( y = \)

(A) \( e^{\tan x} + 4 \)  
(B) \( e^{\tan x} + 5 \)  
(C) \( 5e^{\tan x} \)  
(D) \( \tan x + 5 \)  
(E) \( \tan x + 5e^x \)

\[
\int \frac{dy}{y} = \int \sec^2 x \, dx \\
\ln|y| = \tan x + C \\
y = Ae^{\tan x}
\]

If \( y = 5 \) when \( x = 0 \)

\[
5 = Ae^{\tan 0} \\
5 = A \\
y = 5e^{\tan x}
\]

Choice (C)
4. 1988 BC 43

Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

\( \text{(A) } \frac{3 \ln 3}{\ln 2} \quad \text{(B) } \frac{2 \ln 3}{\ln 2} \quad \text{(C) } \frac{\ln 3}{\ln 2} \quad \text{(D) } \ln \left( \frac{27}{2} \right) \quad \text{(E) } \ln \left( \frac{9}{2} \right) \)

\[
\frac{db}{dt} = kb
\]

\[
\int \frac{db}{b} = \int k \, dt
\]

\[
\ln b = kt + C
\]

\[
b = Ae^{kt}
\]

\[
b(0) = A, \quad b(3) = 2A
\]

Find \( t \) where \( b(t) = 3A \)

\[
2A = Ae^{kt}
\]

\[
2 = e^{3k}
\]

\[
\ln 2 = 3k
\]

\[
\frac{\ln 2}{3} = k
\]

\[
b(t) = Ae^{\frac{\ln 2}{t}}
\]

\[
3A = Ae^{\frac{\ln(3)}{3}}
\]

\[
3 = e^{\frac{\ln(3)}{3}}
\]

\[
\ln 3 = \frac{\ln 2}{3} t
\]

\[
t = \frac{3 \ln 3}{\ln 2} \quad \text{Choice (A)}
\]
5. 1993 AB 33

If \( \frac{dy}{dx} = 2y^2 \) and if \( y = -1 \) when \( x = 1 \), then when \( x = 2 \), \( y = \)

(A) \(-2/3\)  (B) \(-1/3\)  (C) 0  (D) \(1/3\)  (E) \(2/3\)

\[
\int \frac{dy}{y^2} = \int 2 \, dx
\]

\[
\frac{-1}{y} = 2x + C
\]

If \( y = -1 \) when \( x = 1 \),
\[
1 = 2 + C, \quad C = -1
\]

\[
\frac{-1}{y} = 2x - 1
\]

\[y = \frac{-1}{2x - 1}\]

\[y(2) = -\frac{1}{3}\]

Choice (B)
A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

(A) 4.2 pounds  (B) 4.6 pounds  (C) 4.8 pounds
(D) 5.6 pounds  (E) 6.5 pounds

\[
\begin{align*}
\frac{dw}{dt} &= kw \text{ for } 0 \leq t \leq 6 \\
\ln w &= kt + C
\end{align*}
\]

\[
\begin{align*}
w(0) &= 2, \quad w(2) = 3.5 \\
2 &= Ae^{k \cdot 0}, \quad \therefore A = 2 \\
3.5 &= 2e^{2k} \\
\frac{7}{4} &= e^{2k} \\
\frac{1}{2} \ln \frac{7}{4} &= k \\
w &= 2e^{k t} \\
w(3) &= 2 \cdot \left( \frac{7}{4} \right)^{\frac{3}{2}} \approx 4.63 \text{ lbs}
\]

Choice (B)
If $dy/dx = x^2y$, then $y$ could be

(A) $3 \ln \left( \frac{x}{3} \right)$  
(B) $e^{\frac{x^3}{3}} + 7$  
(C) $2e^{\frac{x^3}{3}}$  
(D) $3e^{2x}$  
(E) $\frac{x^3}{3} + 1$

\[ \int \frac{dy}{y} = \int x^2 \, dx \]
\[ \ln |y| = \frac{x^3}{3} + C \]
\[ e^{\ln |y|} = e^{\frac{x^3}{3} + C} \]
\[ |y| = e^{\frac{x^3}{3}} \cdot e^C \]

So $y = Ae^{\frac{x^3}{3}}$, where $A = y(0)$
Choice (C)
During a certain epidemic, the number of people that are infected at any time increases at a rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

(A) 343    (B) 1,343    (C) 1,367    (D) 1,400    (E) 2,057

Let \( p \) = the\ number\ of\ people\ infected\ at\ time\ \( t \).

\[
\frac{dp}{dt} = kp
\]

\[
\int \frac{dp}{p} = \int k \, dt
\]

\[\ln p = kt + C\]

\[p = Ae^{kt}\]

\[p(0) = A = 1000\]

\[p = 1000e^{kt}\]

\[p(7) = 1200 = 1000e^{k \cdot 7}\]

\[\frac{\ln \frac{6}{5}}{7} = k\]

\[p(12) = 1000 \left( \frac{\frac{6}{5}}{7} \right)^{12} \approx 1366.908\]

or 1,367 people    Choice (C)
Based on a number line analysis:

Choice (E)

The graph of the derivative of $f$ is shown in the figure above. Which of the following could be the graph of $f$?

(A)  
(B)  
(C)  
(D)  
(E)
10. 1997 BC 83 (calculator allowed)

If \( \frac{dy}{dx} = (1 + \ln x)y \) and if \( y = 1 \) when \( x = 1 \), then \( y = \)

(A) \( \frac{x^2-1}{x^2} e^{\frac{x^2-1}{x^2}} \)  
(B) \( 1 + \ln x \)  
(C) \( \ln x \)  
(D) \( e^{2x + x\ln x - 2} \)  
(E) \( e^{x\ln x} \)

Note: This question requires a student either to know \( \int \ln x \, dx \) or to be able to use integration by parts to find it.

\[
\int \frac{dy}{y} = \int (1 + \ln x) \, dx \\
\ln |y| = x + x \ln x - x + C \\
\ln |y| = x \ln x + C
\]

Choice (E)

\[
\ln |l| = 1 \cdot \ln(1) + C \\
0 = C \\
\ln y = x \ln x \\
y = e^{x\ln x}
\]
If $\frac{dy}{dt} = ky$ and $k$ is a nonzero constant, then $y$ could be

(A) $2e^{kty}$  (B) $2e^{kt}$  (C) $e^{kt} + 3$  (D) $kty + 5$  (E) $\frac{1}{2}ky^2 + \frac{1}{2}$

\[
\int \frac{dy}{y} = \int k \, dt \\
\ln |y| = kt + C \\
y = Ae^{kt} \text{ and } A \text{ "could be" } 2 \quad \text{Choice (B)}
12. 1998 AB 84 (calculator allowed)

Population $y$ grows according to the equation $\frac{dy}{dt} = ky$, where $k$ is a constant and $t$ is measured in years. If the population doubles every 10 years, then the value of $k$ is

(A) 0.069  (B) 0.200  (C) 0.301  (D) 3.322  (E) 5.000

$y = y(0)e^{kt}$ and if the population doubles every 10 years

$2y(0) = y(0)e^{k\cdot 10}$

$2 = e^{10k}$

$\ln(2) = 10k$ or $k = \ln(2)/10 \approx 0.069$  Choice (A)
If \( \frac{dy}{dx} = \sin x \cos^2 x \) and if \( y = 0 \) when \( x = \pi/2 \), what is the value of \( y \) when \( x = 0 \)?

(A) \(-1\)  (B) \(-1/3\)  (C) 0  (D) 1/3  (E) 1

\[
\int dy = \int \sin x \cos^2 x \, dx
\]
\[
y = -\frac{\cos^3 x}{3} + C
\]
\[
0 = -\frac{\cos^3 \left(\frac{\pi}{2}\right)}{3} + C
\]
\[
0 = C
\]
\[
y(0) = -\frac{1}{3}
\]

Choice (B)
There are numerous ways to approach this problem. One popular student method is the process of elimination:

(A) \( \frac{dy}{dx} = 1 + x \). \( \frac{dy}{dx} = 0 \) when \( x = -1 \). This is not true for this slope field.

(B) \( \frac{dy}{dx} = x^2 \). \( \frac{dy}{dx} \geq 0 \). Some slopes are negative in this slope field.

(C) \( \frac{dy}{dx} = x + y \). \( \frac{dy}{dx} = 0 \) when \( y = -x \). This does match the slope field.

(D) \( \frac{dy}{dx} = x/y \). This would have undefined slopes for \( y = 0 \), the \( x \)-axis.

(E) \( \frac{dy}{dx} = \ln y \). Slopes exist only for \( y > 0 \).

Therefore, the answer must be Choice (C).

Another approach is to realize that if \( \frac{dy}{dx} \) is a function of \( x \) only, then the slope field looks like columns of parallel segments. If \( \frac{dy}{dx} \) is a function of \( y \) only, then the slope field looks like rows of parallel segments. Since this is not a feature of the slope field, (A), (B), and (E) are eliminated at a glance. (D) is eliminated because slopes in quadrants I and III would be positive.

Shown above is a slope field for which of the following differential equations?

(A) \( \frac{dy}{dx} = 1+x \)  
(B) \( \frac{dy}{dx} = x^2 \)  
(C) \( \frac{dy}{dx} = x+y \)  
(D) \( \frac{dy}{dx} = \frac{x}{y} \)  
(E) \( \frac{dy}{dx} = \ln y \)
15. 1998 BC 26

The population $P(t)$ of a species satisfies the logistic differential equation,

$$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$$

where the initial population $P(0) = 3,000$ and $t$ is the time in years. What is $\lim_{t \to \infty} P(t)$?

(A) 2,500      (B) 3,000      (C) 4,200      (D) 5,000      (E) 10,000

$$\frac{dP}{dt} = P\left(\frac{10,000 - P}{5000}\right) = 0$$

when $P = 0$ or $P = 10,000$ (the carrying capacity).

$$\lim_{t \to \infty} P(t) = 10,000$$

(It reaches no change at its carrying capacity.) Choice (E)