Modeling Log Functions

DAY 7

Use the compound interest formulas and your calculator to answer each of the following questions. Round your answer to the nearest cent.

$n=4$

1. \$100 is invested into an account that pays 4% annual interest compounded quarterly. How much is in the account after 2 years?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 100\left(1+\frac{0.04}{4}\right)^{4(2)}$$

$$\boxed{A \approx \$108.29}$$

2. How much do you need to initially deposit in order to get \$100 after 2 years at 6% compounded monthly?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$100 = P\left(1+\frac{0.06}{12}\right)^{12(2)}$$

$$\frac{100}{1.127} = \frac{P(1.127)}{1.127}$$

$$\boxed{P \approx \$88.72}$$

3. How long will it take your initial investment to triple in an account that pays 12% annual interest compounded annually?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$3 = 1\left(1+\frac{0.12}{1}\right)^{1\cdot t}$$

$$\ln 3 = \ln(1+0.12)^t$$

$$\ln 3 = t(\ln 1.12)$$

$$t = \frac{\ln 3}{\ln 1.12}$$

$$\boxed{t \approx 9.69 \text{ years}}$$

4. How long will it take for an initial investment of \$25,000 to grow to \$80,000 in an account that pays 7% annual interest compounded continually?

$$A = Pe^{rt}$$

$$\frac{80{,}000}{25{,}000} = \frac{25{,}000e^{0.07t}}{25{,}000}$$

$$3.2 = e^{0.07t}$$

$$\ln 3.2 = \ln e^{0.07t}$$

$$\frac{\ln 3.2}{0.07} = \frac{0.07t}{0.07}$$

$$\boxed{t \approx 16.62 \text{ years}}$$

MODELING LOG FUNCTIONS

Today, we will be answering questions using the function $A = Pe^{rt}$, where r is the rate of change (growth or decay). A wide variety of situations can be modeled by this exponential function, including the ones we will study

- Population change
- Half-life decay
- Carbon Dating
- Newton's Law of Cooling

1. **"The Population Problem"**
In 1990, the population of Loudoun County was 87,205 people. 10 years later, in 2000, the population of Loudoun County increased to 173,897 people. Predict what the population was in 2010. What about 2013?

$$\frac{173,897}{87,205} = \frac{87,205e^{r(10)}}{87,205}$$

$$\ln 1.99 = \ln e^{10r}$$

$$\frac{\ln 1.99}{10} = \frac{10r}{10}$$

$$r \approx 0.069$$

$$\boxed{\text{Population Rate} = 6.8\%}$$

2. The population of the United States was approximately 226.5 million people in 1980. In 1995, the population of the United States was 266.3 million people. Predict what the population was in 2010.

$$\frac{266.3}{226.5} = \frac{226.5e^{r(15)}}{226.5}$$

$$\ln 1.18 = \ln e^{15r}$$

$$\frac{\ln 1.18}{15} = \frac{15r}{15}$$

$$0.011 \approx r \rightarrow x$$

$$A = Pe^{rt}$$

$$A = Pe^{xt}$$

$$A = 226.5e^{x(30)}$$

$$\boxed{A \approx 315.379 \text{ million People}}$$

3. A biologist is studying a new strain of bacteria. At time t = 0, he places a sample of 100 bacteria into a Petri dish to measure its growth rate. After only 6 hours, the biologist measures 250 bacteria. Assuming an exponential growth, at what rate was the bacteria growing? How long will it take the bacteria to triple? How much bacteria will be there after 12 hours? How long before we have 2000 bacteria?

$A = Pe^{rt}$

$\frac{250}{100} = \frac{100e^{r(6)}}{100}$

$\ln 2.5 = \ln e^{6r}$

$\frac{\ln 2.5}{6} = \frac{6r}{6}$

$r = \frac{\ln 2.5}{6} \approx 0.153 \to x$

$\boxed{r \approx 15.3\%}$

$\frac{300}{100} = \frac{100e^{xt}}{100}$

$\ln 3 = \ln e^{xt}$

$\frac{\ln 3}{x} = \frac{xt}{x}$

$\boxed{t \approx 7.19 \text{ hrs}}$

$A = 100e^{x(12)}$

$\boxed{A = 625 \text{ bacteria}}$

$\frac{2000}{100} = \frac{100e^{xt}}{100}$

$\ln 20 = \ln e^{xt}$

$\frac{\ln 20}{x} = \frac{xt}{x}$

$\boxed{t \approx 19.62 \text{ hours}}$

4. **"The Half-life Problem"**
The half-life of Magnesium-27 is 9.45 minutes. How long will it take for an initial sample of 50 grams to decay to a sample of 10 grams?

$A = Pe^{rt}$

$\frac{25}{50} = \frac{50e^{r(9.45)}}{50}$

$\ln \frac{1}{2} = \ln e^{9.45r}$

$\frac{\ln \frac{1}{2}}{9.45} = r \to x$

$A = Pe^{xt}$

$\frac{10}{50} = \frac{50e^{xt}}{50}$

$\ln \frac{1}{5} = \ln e^{xt}$

$\frac{\ln \frac{1}{5}}{x} = t$ $\boxed{t \approx 21.94 \text{ min}}$

5. Some people are frightened of certain medical tests because the tests involve the injection of radioactive materials. A hepatobiliary scan of a gallbladder involves an injection of 0.5 cc's of Technetium-99m, which has a half-life of almost exactly 6 hours. While undergoing the test, the technician says that after twenty-four hours, you'll be down to background radiation levels. Figure out just how much radioactive material remained from the gallbladder scan after twenty-four hours.

$A = Pe^{rt}$

$\ln \frac{1}{2} = \ln e^{r(6)}$

$\frac{\ln \frac{1}{2}}{6} = \frac{6r}{6}$

$\frac{\ln \frac{1}{2}}{6} = r \to x$

$A = Pe^{xt}$

$A = 0.5e^{x(24)}$

$\boxed{A = 0.03125 \text{ cc}}$

7. **"The Carbon Dating Problem"**
Carbon-14 is used to date old artifacts as it has a large half-life of 5,730 years. You find an animal bone that has lost 18% of its carbon – 14. How old is the animal bone? A second bone is found that has lost 23% of its carbon-14. How old is the second bone?

$A = Pe^{rt}$

$\ln \frac{1}{2} = \ln e^{r(5730)}$

$\ln \frac{1}{2} = 5730r$

$\frac{\ln \frac{1}{2}}{5730} = r \rightarrow x$

$0.82 = e^{xt}$

$\ln 0.82 = xt$

$\frac{\ln 0.82}{x} = t$

$\boxed{t \approx 1640.53 \text{yrs}}$

$0.77 = e^{xt}$

$\frac{\ln 0.77}{x} = \frac{xt}{x}$

$\frac{\ln 0.77}{x} = t$

$\boxed{t \approx 2160.61 \text{yrs}}$

8. **"Car Stopping Problem"**
Tim is driving along a straight highway at 64 km/hr when the car runs out of gas. As he slows down, his speed decreases exponentially with the number of seconds since he ran out of gas, dropping to 48 km/hr after 10 seconds. How long after breaking is Tim traveling half his original speed?

$A = Pe^{rt}$

$\frac{48}{64} = \frac{64e^{r(10)}}{64}$

$\ln 0.75 = \ln e^{10r}$

$\frac{\ln 0.75}{10} = r \rightarrow x$

$A = Pe^{xt}$

$\ln \frac{1}{2} = \ln e^{xt}$

$\frac{\ln \frac{1}{2}}{x} = \frac{xt}{x}$

$t = \frac{\ln \frac{1}{2}}{x}$

$\boxed{t \approx 24.09 \text{ sec}}$

9. **"Newton's Law of Cooling"**
When you pour a cup of coffee, it cools in such a way that the difference between the coffee temperature and the room temperature decreases exponentially with time. Suppose three minutes after pouring a cup of coffee, the temperature of the coffee is 85˚C. Then, 5 minutes after that the temperature is found to be 72˚C. If the room is 20˚C, when will the coffee be cool enough to drink, 55˚C.

$T(t) = Pe^{rt} + T_E$

$85 = Pe^{3r} + 20$

$\frac{65}{e^{3r}} = \frac{Pe^{3r}}{e^{3r}}$

$\frac{65}{e^{3r}} = P$

$72 = Pe^{8r} + 20$

$\frac{52}{e^{8r}} = \frac{Pe^{8r}}{e^{8r}}$

$\frac{52}{e^{8r}} = P$

$\frac{65}{e^{3r}} = \frac{52}{e^{8r}}$

$\frac{65e^{8r}}{e^{3r}} = \frac{52e^{3r}}{e^{3r}}$

$\frac{65e^{5r}}{65} = \frac{52}{65}$

$\ln e^{5r} = \ln 0.8$

$5r = \ln 0.8$

$\boxed{r \approx -0.446}$