<table>
<thead>
<tr>
<th>Riemann Sums Given a Function</th>
<th>TURN AND TALK: Given a strictly increasing or decreasing function, can you tell whether the endpoint you choose will yield an over-approximation or under-approximation?</th>
</tr>
</thead>
</table>

**Left endpoints:** under approx  
**Right endpoints:** over approx  

How do we make the approximation better?

Area under a curve: This is called a _________________.

**EXAMPLE:** Approximate the area bounded above by the graph of \( f(x) = x^2 + 1 \) and below by the \( x \)-axis on the interval \([0, 1]\) with using Riemann sums with 4 subintervals. We'll do left, right, and midpoint sums.

1. Find \( \Delta x \)
2. Make a table
3. Write out areas of rectangles
Riemann Sums
Given a Table

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>12</td>
<td>28</td>
<td>34</td>
<td>30</td>
</tr>
</tbody>
</table>

The function \( f \) is continuous on the closed interval \([2, 14]\) and has values shown in the table above. Using the subintervals \([2, 5]\), \([5, 10]\), and \([10, 14]\), what is the approximation for the area between \( f(x) \) and the \( x \)-axis on the interval \([2, 14]\) found by using a right Riemann sum?

(A) 296    (B) 312    (C) 343    (D) 374    (E) 390

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(t) ) (liters / hour)</td>
<td>1340</td>
<td>1190</td>
<td>950</td>
<td>740</td>
<td>700</td>
</tr>
</tbody>
</table>

Water is removed from a tank at a rate modeled by \( R(t) \) liters per hour, where \( R \) is differentiable and decreasing \( 0 \leq t \leq 8 \), where \( t \) is measured in hours. Selected values of \( R(t) \) are shown in the table above. At time \( t = 0 \), there are 50,000 liters of water in the tank.

Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours.

\[
1(1340) + 2(1190) + 3(950) + 2(740) = 8050 \text{ L}
\]

Is this an overestimate or underestimate of the total amount of water removed? Give a reason for your answer.

**Overestimate** because \( R(t) \) is decreasing

What is the total amount of water left in the tank after 8 hours?

\[
50000 - 8050 = 41,950 \text{ L}
\]
Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time $t$, $0 \leq t \leq 6$, is given by a differentiable function $C$, where $t$ is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

a) Use the data in the table to approximate $C'(3.5)$. Indicate units of measure.

b) Is there a time $t$, $2 \leq t \leq 4$, at which $C'(t) = 2$? Justify your answer.

c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the amount of coffee that fills the cup in the first six minutes.

$$2(5.3) + 2(11.2) + 2(13.8) = 60.6$$
Notes: 7.2 Trapezoidal Rule

Trapezoidal Rule

Rectangles are ok, but trapezoids are better!

Area of a trapezoid:
\[ h\left(\frac{b_1+b_2}{2}\right) = \frac{1}{2} h(b_1+b_2) \]

Area of ith trapezoid:
\[ \frac{1}{2} \Delta x (f(x_i) + f(x_{i+1})) \]

EXAMPLE: Use Trapezoidal Rule to approximate the area bounded by the graph of \( y = \sin x \) and the x-axis from \( x = 0 \) to \( x = \pi \) for \( n = 4 \).

\[ \Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{3\pi}{4} )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>1</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \frac{1}{2} \left( \frac{\pi}{4} \right) \left( 0 + \frac{\sqrt{2}}{2} \right) + \frac{1}{2} \left( \frac{\pi}{4} \right) \left( \frac{\sqrt{2}}{2} + 1 \right) + \frac{1}{2} \left( \frac{\pi}{4} \right) \left( 1 + \frac{\sqrt{2}}{2} \right) + \frac{1}{2} \left( \frac{\pi}{4} \right) \left( \frac{\sqrt{2}}{2} + 0 \right) \]

\[ \frac{1}{8} \left( 2 \sqrt{2} + 2 \right) = 1.896 \]

Over and Under Approximations

When is a trapezoidal rule an over-approximation? An under-approximation?

- concave up: over approx
- concave down: under approx
A table of values for a continuous function $f$ is shown to the right. If four equal subintervals of $[0, 2]$ are used, which of the following is the trapezoidal approximation of the area under $f$ on the interval $[0, 2]$.

\[
\begin{align*}
\Delta x &= 0.5 \\
\text{a)} \quad 8 \\
\text{b)} \quad 12 \\
\text{c)} \quad 16 \\
\text{d)} \quad 24 \\
\text{e)} \quad 32
\end{align*}
\]

\[
\frac{1}{2}(0.5)(3+3) + \frac{1}{2}(0.5)(3+5) + \frac{1}{2}(0.5)(5+8) + \frac{1}{2}(0.5)(8+13) = 12
\]

Find the change in position of the train (displacement) from time $t = 2$ to time $t = 12$ using a trapezoidal sum with three subintervals.

\[
\begin{array}{c|c|c|c|c|c}
\text{t (minutes)} & 0 & 2 & 5 & 8 & 12 \\
\hline
v_0(t) \text{ (meters/minute)} & 0 & 100 & 40 & -120 & -150 \\
\end{array}
\]

\[
\frac{1}{2}(3)(100+40) + \frac{1}{2}(3)(40+(-120)) + \frac{1}{2}(4)((-120)+(-150)) = -450 \text{ m} \Rightarrow 450 \text{ m behind start}
\]

The sketch at the right shows a lake that is 700 feet long from top to bottom. The width of the lake is measured at 100 foot intervals; these measurements, in feet, are marked on the sketch (not to scale).

Approximate the area of the surface of the lake. Indicate units of measure.

\[
\frac{1}{2}(100) \left[ (0+450) + (450+500) \right. \\
\left. + (500+450)+(450+420) \\
\left. + (420+560)+(560+580) \\
\left. + (580+0) \right] = 293,000 \text{ ft}^2
\]