Using a Sign Chart

Find the relative extrema of the function \( f(x) = \frac{1}{2}x - \sin x \) on \((0, 2\pi)\).

<table>
<thead>
<tr>
<th>Intervals</th>
<th>Sign of ( f' )</th>
<th>Behavior of ( f )</th>
</tr>
</thead>
</table>

\[
\begin{align*}
\text{Find the relative extrema of } f(x) &= (x^2 - 4)^{\frac{1}{3}}. \\
\frac{d}{dx} f(x) &= \frac{2}{3} (x^2 - 4)^{-\frac{1}{3}} (2x) \\
f'(x) &= \frac{4x}{3 (x^2 - 4)^{\frac{2}{3}}} = 0 \\
f(x) &= 0 \\
4x &= 0 \\
x &= 0 < \text{critical value}
\end{align*}
\]

\[
\text{On what intervals is } f(x) \text{ increasing?} \\
(-2, 0) \cup (2, \infty) \quad \text{because } f' > 0
\]

\[
\text{On what intervals is } f(x) \text{ decreasing?} \\
(-\infty, -2) \cup (0, 2) \quad \text{because } f' < 0
\]
Find the relative extrema of \( f(x) = \frac{x^4 + 1}{x^2} = (x^4 + 1)x^{-2} \):

\[
f(x) = \frac{x^4}{x^2} + \frac{1}{x^2} = x^2 + x^{-2}
\]

\[
f'(x) = 2x - 2x^{-3}
\]

\[
0 = 2x - \frac{2}{x^3}
\]

\[
\frac{2}{x^3} = 2x = \frac{2x^4}{x^4} = 1
\]

\[
x = \pm 1
\]

The function \( f \) given by \( f(x) = x^3 + 12x - 24 \) is

(A) increasing for \(-2 < x < -1\), decreasing for \(-1 < x < 2\), increasing for \( x > 2 \)

(B) decreasing for \( x < 0 \), increasing for \( x > 0 \)

(C) increasing for all \( x \) because \( f' \) is always \( > 0 \)

(D) decreasing for all \( x \)

(E) decreasing for \( x < -2, x > 2 \), increasing for \( -2 < x < 2 \)

\[
f'(x) = 3x^2 + 12
\]

\[
0 = 3x^2 + 12
\]

\[
-12 = 3x^2
\]

\[
-4 = x^2
\]

no critical numbers!

The function defined by \( f(x) = x^3 - 3x^2 \) for all real numbers \( x \) has a relative maximum at \( x = \)

\[
x_1 = -2
\]

\[
\text{(B) 0}
\]

\[
x_2 = 1
\]

\[
\text{(D) 2}
\]

\[
x_3 = 4
\]

\[
f'(x) = 3x^2 - 6x
\]

\[
0 = 3x(x - 2)
\]

\[
x = 0\quad x = 2
\]
Notes: 5.4 Connecting $f'$ and $f''$ with the Graph of $f$
Concavity

<table>
<thead>
<tr>
<th>Definition of Concavity</th>
<th>The graph of $f$ is concave upward if $f'$ is increasing.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>The graph of $f$ is concave downward if $f'$ is decreasing.</td>
</tr>
<tr>
<td>Pictures:</td>
<td><img src="image" alt="Concave Graph" /></td>
</tr>
<tr>
<td></td>
<td>Concave up like a cup</td>
</tr>
<tr>
<td></td>
<td>$m = 0$</td>
</tr>
<tr>
<td></td>
<td>$m = 1$</td>
</tr>
<tr>
<td></td>
<td>Concave down like a frown</td>
</tr>
<tr>
<td></td>
<td>$m = -1$</td>
</tr>
<tr>
<td></td>
<td>$m = -2$</td>
</tr>
</tbody>
</table>

Test for Concavity
We can figure out the concavity of a graph by looking at the sign of the second derivative $(f'')$.

If $f''(x) > 0$, then the graph is concave up.

If $f''(x) < 0$, then the graph is concave down.

Determining Concavity
Determine the open intervals on which the graph of $f(x) = \frac{6}{x^2 + 3}$ is concave upward or downward.

\[
f(x) = 6(x^2 + 3)^{-1}
\]
\[
f'(x) = -12x(x^2 + 3)^{-2}(2x) = -\frac{12x}{(x^2 + 3)^2}
\]
\[
f''(x) = -12x[-2(x^2 + 3)^{-3}(2x)] + (x^2 + 3)^{-2}(-12x) = \frac{48x^2 - 12x^2 - 36}{(x^2 + 3)^3} = \frac{36x^2 - 36}{(x^2 + 3)^3}
\]
\[
= 0 = 36x^2(x^2 - 1)
\]
\[
x = \pm 1
\]
\[
\begin{array}{cccc}
+ & - & - & + \\
+ & - & + & + \\
+ & - & + & + \\
\end{array} 
\]
\[
f''
\]

Concave up: $(-\infty, -1) \cup (1, \infty)$ because $f'' > 0$

Concave down: $(-1, 1)$ because $f'' < 0$
A point of inflection is where $f''$ changes signs

$(f' \text{ changes from inc} \rightarrow \text{dec or dec} \rightarrow \text{inc})$

CONCAVITY CHANGES

If $(c, f(c))$ is a point of inflection, then $f''(c) = 0$ or is undefined.

Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$.

$f'(x) = 4x^2 - 12x^2$
$f''(x) = 12x^2 - 24x$
$0 = 12x(x - 2)$
$x = 0 \quad x = 2$

$f$ has a point of inflection at $x = 0$ and at $x = 2$ because $f''$ changes sign.

$f$ is concave up on $(-\infty, 0) \cup (2, \infty)$ because $f'' > 0$.

$f$ is concave down on $(0, 2)$ because $f'' < 0$. 