Notes: 5.2 Mean Value Theorem

Mean Value Theorem:

If $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there is a $c$ between $a$ and $b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Picture:

How would we find $c$?

- Take a derivative
- Find $\frac{f(b) - f(a)}{b - a}$
- Set equal & solve for $x$-value

Given $f(x) = 5 - \frac{4}{x}$, show that there is a point $c$ in the interval $(1, 4)$ where the average rate of change equals the instantaneous rate of change. Then, find the value of $c$ in the interval where this happens.

$f(x)$ is continuous and differentiable on $(1, 4)$.

$\frac{f(4) - f(1)}{4 - 1} = \frac{(5 - \frac{4}{4}) - (5 - \frac{4}{1})}{3} = \frac{4 - 1}{3} = 1$

Conclusion:

By MVT there is a $c$ between 1 and 4 where $f'(c) = 1$.

Example (given a function)

Example (how to break the theorem)

Function should be continuous & differentiable.

$f(x) = |x| + 1$

$f(x)$ is not differentiable at $x = 0$.

$g(x) = \begin{cases} x^3 + 3 & \text{when } x < 1 \\ x^2 - 1 & \text{when } x \geq 1 \end{cases}$

$g(x)$ is not continuous at $x = 1$. 

Not in interval
2017 #6d: is there a number \( c \) in the closed interval \([-5, -3]\) such that \( g'(c) = -4? \) Justify your answer.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>10</td>
<td>-3</td>
</tr>
<tr>
<td>-4</td>
<td>5</td>
<td>-1</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

\[ g(x) \text{ is differentiable therefore }\]
\[ g(x) \text{ is continuous} \]

\[ g(-3) - g(-5) = \frac{2 - 10}{-3 + 5} = \frac{-8}{2} = -4 \]

By MVT there is an \( c \) between \(-5 \leq c \leq -3\) where \( g'(c) = -4 \)

2013 #3b: Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time \( t \), \( 0 \leq t \leq 6\), is given by a differentiable function \( C\), where \( t \) is measured in minutes. Selected values of \( C(t)\), measured in ounces, are given in the table.

Is there a time \( \frac{3}{4} \leq t \leq \frac{5}{4} \) at which \( C'(t) = 2? \) Justify your answer.

<table>
<thead>
<tr>
<th>( t ) (minutes)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(t) ) (ounces)</td>
<td>0</td>
<td>5.3</td>
<td>8.8</td>
<td>11.2</td>
<td>12.8</td>
<td>13.8</td>
<td>14.5</td>
</tr>
</tbody>
</table>

\( C(t) \) is differentiable therefore \( C(t) \) is continuous

\[ \frac{C(4) - C(2)}{4 - 2} = \frac{12.8 - 8.8}{2} = \frac{4}{2} = 2 \]

By MVT there's a time between \( 2 \) and \( 4 \) where \( C'(t) = 2 \)