Related Rates Practice

Let \( A \) be the area of a square whose sides have length \( x \) and assume that \( x \) varies with time. At a certain instant, the sides are 3 ft long and growing at a rate of 2 ft/min. How fast is the area growing at that instant?

\[
\frac{dA}{dt} = 2x \frac{dx}{dt}
\]
\[
\frac{dA}{dt} = 2(3)(2)
\]
\[
\frac{dA}{dt} = 12
\]

The area is growing at 12 ft\(^2\)/min.

Given \( \frac{dx}{dt} = 2 \text{ ft/min} \) when \( x = 3 \)

Find \( \frac{dA}{dt} \)

2. Let \( V \) be the volume of a cylinder having height \( h \) and radius \( r \). Assume \( h \) and \( r \) vary with time. At a certain instant, the height is 6 inches and is increasing by 1 in/sec, while the radius is 10 in. and decreasing by 1 in/sec. How fast is the volume changing at that instant? (\( V_{cylinder} = \pi r^2 h \))

\[
V = \pi r^2 h
\]
\[
\frac{dV}{dt} = \pi \left[ 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right]
\]
\[
\frac{dV}{dt} = \pi \left[ (10^2)(1) + (6)(2)(10)(-1) \right]
\]
\[
\frac{dV}{dt} = -20\pi
\]

The volume is decreasing at a rate of 20\pi in\(^3\)/sec.

Given \( \frac{dh}{dt} = 1 \text{ in/sec} \)

Find \( \frac{dV}{dt} \)

when \( h = 6 \) and \( r = 10 \)

3. A spherical balloon is inflated so that its volume is increasing at a rate of 3 ft\(^3\)/min. How fast is the diameter of the balloon increasing when the radius is 1 foot?

\[
V = \frac{4}{3} \pi r^3
\]
\[
\frac{dV}{dt} = 4\pi \frac{d}{dt} \left( \frac{r^3}{3} \right)
\]
\[
\frac{dV}{dt} = \frac{4}{3} \pi \frac{d}{dt} r^3
\]
\[
\frac{dV}{dt} = \frac{4}{3} \pi \frac{dr}{dt} \cdot \frac{d}{dt} r^3
\]
\[
\frac{dV}{dt} = \frac{4}{3} \pi \frac{dr}{dt} (3r^2)
\]
\[
\frac{dV}{dt} = \frac{2}{3} \pi \left( \frac{r^3}{2} \right)
\]
\[
\frac{dV}{dt} = \frac{3}{2\pi}
\]
\[
\frac{dV}{dt} = \frac{3}{2\pi}
\]
\[
\frac{dV}{dt} = \frac{3}{2\pi}
\]
\[
\frac{d}{dt} = \frac{2}{\pi^2} \frac{dV}{dt}
\]
\[
\frac{d}{dt} = \frac{2}{\pi^2} \cdot 0.48
\]
\[
\frac{d}{dt} = 0.48 \text{ ft/min}
\]

When \( r = 1 \), \( \frac{dr}{dt} = \frac{3}{2\pi} \)

The diameter is increasing at about 0.48 ft/min.
4. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 \text{ mi}^2/\text{hr}. How fast is the radius of the spill increasing when the area is 9 \text{ mi}^2?

\[ \frac{dA}{dt} = 6 \text{ mi}^2/\text{hr} \]

Find \( \frac{dr}{dt} \) when \( A = 9 \text{ mi}^2 \)

\[ \frac{dr}{dt} = \frac{1}{2\pi r} \cdot \frac{dA}{dt} \]

5. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm? \( (V_{\text{sphere}} = \frac{4}{3}\pi r^3) \)

\[ V = \frac{4}{3}\pi r^3 \]

\[ \frac{dV}{dt} = \frac{4}{3} \pi r^2 \frac{dr}{dt} \]

\[ \frac{dV}{dt} = 4\pi r^2 (9)(-15) \]

\[ \frac{dV}{dt} = -4860\pi \]

6. A 13 foot ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/sec, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

\[ x^2 + y^2 = 13^2 \]

\[ \frac{d}{dt}[x^2 + y^2] = \frac{d}{dt}[13]^2 \]

\[ 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \]

\[ \frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} \]

\[ \frac{dx}{dt} = -\left(\frac{5}{12}\right) (-2) \]

\[ \frac{dx}{dt} = \frac{5}{6} \approx 0.83 \text{ ft/sec} \]
7. At a certain instant each edge of a cube is 5 in long and the volume is increasing at the rate of 2 in$^3$/min. How fast is the surface area of the cube increasing? ($V_{cube} = s^3; \text{SA}_{cube} = 6s^2.$)

\[
\frac{dV}{dt} = 3s^2 \frac{ds}{dt} \quad \frac{dA}{dt} = 12s \frac{ds}{dt} \\
\frac{ds}{dt} = \frac{1}{3} \frac{dV}{dt} = \frac{1}{3} \cdot \frac{2}{15} \quad \frac{dA}{dt} = 12(5) \left(\frac{2}{15}\right) \\
\frac{ds}{dt} = \frac{2}{15} \quad \frac{dA}{dt} = \frac{24}{15} = \frac{8}{5} = 1.6 \\
\]

The S.A. is increasing at a rate of 1.6 in/min.

\[\text{Given } \frac{dV}{dt} = 2 \text{ in}^3/\text{min} \quad \text{when } s = 5\]

Find \(\frac{dA}{dt}\)

8. For the camera shown in the figure below, at what rate is the elevation angle changing when the rocket is 3000 ft. up and rising vertically at 500 ft/sec?

\[
\tan \theta = \frac{y}{x} \\
\frac{d}{dt}[\tan \theta] = \frac{d}{dt}\left[\frac{y}{x}\right] \\
\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{3000} \frac{dy}{dt} \\
\frac{d\theta}{dt} = \frac{\sec \theta}{(\sec \theta)^2} \cdot \frac{1}{3000} \cdot 500 \\
\frac{d\theta}{dt} = \frac{1}{(\sqrt{2})^2} \cdot \frac{1}{3000} \cdot 500 \\
\frac{d\theta}{dt} = \frac{1}{12} \approx 0.08
\]

The elevation is increasing at a rate of 0.08 rad/sec.

\[\text{Given } \frac{dy}{dt} = 500 \text{ ft/sec} \quad \text{when } y = 3000\]

Find \(\frac{d\theta}{dt}\)

9. Grain pouring from a chute at the rate of 8 ft$^3$/min forms a conical pile whose altitude is always twice its radius. How fast is the altitude of the pile increasing at the instant when the pile is 6 ft. high?

\[V = \frac{1}{3} \pi r^2 h \\
V = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 \cdot h = \frac{1}{12} \pi h^3 \\
\]

\[\frac{dV}{dt} = \frac{3}{12} \pi h^2 \cdot \frac{dh}{dt} \\
\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt} \\
\frac{dh}{dt} = \frac{4}{\pi (6)^2} \cdot (8) \\
\frac{dh}{dt} = \frac{8}{9\pi} \approx 0.28
\]

The pile is increasing at a rate of 0.28 ft/min.

\[\text{Given: } \frac{dV}{dt} = 8 \text{ ft}^3/\text{min} \quad \text{Find } \frac{dh}{dt} \text{ when } h = 6\]
10. The minute hand of a clock is 4 in. long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

\[
A = \frac{1}{2} r^2 \theta = \frac{1}{2} (4)^2 \theta = 8\theta
\]

\[
\frac{dA}{dt} = 8 \frac{d\theta}{dt}
\]

\[
\frac{dA}{dt} = 8 \left( \frac{\pi}{30} \right) = \frac{4\pi}{15} \approx 0.84
\]

The area increases at a rate of 0.84 in\(^2\)/min.

Find \( \frac{dA}{dt} \)

11. A 10 foot plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/sec, how fast is the acute angle that the plank makes with the ground increasing?

\[
\cos \theta = \frac{x}{10}
\]

\[
\frac{d}{dt} \left[ \cos \theta \right] = \frac{dx}{dt} \left[ \frac{1}{10} \right]
\]

\[
-\sin \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt}
\]

\[
\frac{d\theta}{dt} = \left( \frac{-1}{\sin \theta} \right) \left( \frac{1}{10} \right) \frac{dx}{dt}
\]

\[
\frac{d\theta}{dt} = \left( \frac{-1}{2\sqrt{2}} \right) \left( \frac{1}{10} \right) \left( \frac{1}{2} \right) = \frac{1}{8\sqrt{2}} \approx 0.05
\]

\[
\sin \theta = \frac{4\sqrt{2}}{10}
\]

\[
\sin \theta = \frac{2\sqrt{6}}{5}
\]
12. A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet/second.

a. How fast is the top of the ladder moving down the wall when its base is 7 feet from the wall?

\[
\frac{dx}{dt} \left[ x^2 + y^2 \right] - \frac{d}{dt} \left[ 25 \right] = 0
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

\[
\frac{dy}{dt} = \frac{-x \frac{dx}{dt}}{y}
\]

\[
\frac{dy}{dt} = \left( -\frac{7}{24} \right) = -0.58
\]

b. Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

\[
A = \frac{1}{2}bh = \frac{1}{2}xy
\]

\[
\frac{dA}{dt} = \frac{1}{2} \left[ x \frac{dy}{dt} + y \frac{dx}{dt} \right]
\]

\[
\frac{dA}{dt} = \frac{1}{2} \left[ 7 \left( -\frac{7}{12} \right) + 24 \left( \frac{2}{25} \right) \right]
\]

\[
\frac{dA}{dt} = \frac{1}{2} \left( -\frac{49}{12} + \frac{48}{25} \right)
\]

\[
\frac{dA}{dt} \approx 2.2
\]

c. Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

\[
\sin \theta = \frac{x}{25}
\]

\[
\frac{d}{dt} \left[ \sin \theta \right] = \frac{d}{dt} \left[ \frac{x}{25} \right]
\]

\[
\cos \theta \frac{dx}{dt} = \frac{1}{25} \frac{dx}{dt}
\]

\[
\frac{d\theta}{dt} = \left( \frac{1}{\cos \theta} \right) \left( \frac{1}{25} \right) \left( \frac{2}{2} \right)
\]

\[
\frac{d\theta}{dt} = \left( \frac{\sqrt{25}}{31} \right) \left( \frac{1}{25} \right) \left( \frac{2}{2} \right)
\]

\[
\frac{d\theta}{dt} = \frac{1}{12} \approx 0.08 \text{ rad}
\]

The angle \( \theta \) is increasing at a rate of 0.08 rad/sec.