Name: ______________________________ Date: ___________________ Block: ______

One-Sided & Infinite Limits

A general limit is the value or output where a function 'approaches' a specific input. A one-sided limit looks specifically at one side of the input in question. Formally we have the following notation:

Left-Handed Limit: $\lim_{x\to a^-} f(x) = L$

Right-Handed Limit: $\lim_{x\to a^+} f(x) = L$

Example #1 – Given the function $f(x)=\begin{cases} 2x-3 & x\le 2 \\ x^2-1 & x>2 \end{cases}$ determine each of the following:

a) $\lim_{x\to 2^-} f(x) =$ 1

b) $\lim_{x\to 2^+} f(x) =$ 3

c) What does this imply about the $\lim_{x\to 2} f(x)$? Explain. The limit as x approaches 2 does not exist b/c $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

Example #2 – Use one-sided limits to find the value of a so that $\lim_{x\to 2} f(x) = 1$ given that $f(x)=\begin{cases} 2x-3, & x\le 2 \\ x^2+a, & x>2 \end{cases}$

$x^2 + a = 1$ $\quad$ $2x-3=1$

$a=-3$ $\quad$ $4-3=1$

$\therefore \lim_{x\to 2^+} f(x) = 1$ $\quad$ $1=1$ ✓

Example #3 – Estimate the left- and right-handed limits for the function $\cos\left(\frac{\pi}{t}\right)$ at $t=0$.

$\lim_{t\to 0^-} \cos\left(\frac{\pi}{t}\right) =$ dne

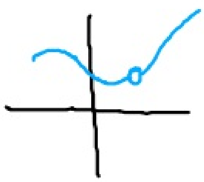
$\lim_{t\to 0^+} \cos\left(\frac{\pi}{t}\right) =$ dne

Example #4 – Use the values in the table to approximate $\lim_{x\to -1.8} f(x)$.

x	−1.83	−1.82	−1.81	−1.8	−1.79	−1.78	−1.77
y	−22.51	−22.54	−22.57	−22.60	−22.63	−22.66	−22.69

AP Multiple Choice

If $\lim_{x\to a} f(x) = L$, where L is a real number, which of the following must be true?



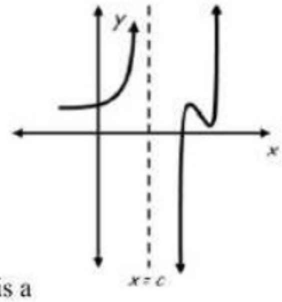
~~I.~~ $f(a) = L$ $\quad$ II. $\lim_{x\to a^-} f(x) = L$ $\quad$ III. $\lim_{x\to a^+} f(x) = L$

~~(A)~~ I only
~~(B)~~ I and II
~~(C)~~ I and III
(D) II and III

Definition: *If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or left, the line $x = c$ is a vertical asymptote of the graph of $f(x)$*

Critical Point

Any x-value that makes a function undefined will automatically be a vertical asymptote. This will usually be where the denominator is zero for a rational function.



Formally if either $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$, then the line $x = a$ is a *vertical asymptote* of the graph of the function $y = f(x)$.

Example #5 – Find the limit for the following.

a) $\lim\limits_{x\to 3^+} \dfrac{x^2}{x^2-9} = \infty$

VA:
$x^2 - 9 = 0$
$x^2 = 9$
$x = \pm 3$

$\dfrac{3.1^2}{3.1^2 - 9} = \dfrac{pos}{pos} > 0$

b) $\lim\limits_{x\to -1^-} \dfrac{x^2+6x-7}{x+1} = \infty$

$\lim\limits_{x\to -1^-} \dfrac{(x+7)(x-1)}{x+1}$

VA:
$x + 1 = 0$
$x = -1$

$\dfrac{(-1.1+7)(-1.1-1)}{-1.1+1}$

$\dfrac{pos \cdot neg}{neg} = \dfrac{neg}{neg} > 0$

c) $\lim\limits_{x\to 0^+} \dfrac{2}{\sin x} = \infty$

VA:
$\sin x = 0$
$x = 0, \pi$

$\dfrac{2}{\sin \frac{\pi}{6}} = \dfrac{2}{pos} > 0$

Properties of Infinite Limits

Let f and g be functions such that $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = L$.

1. $\lim\limits_{x\to c}[f(x) \pm g(x)] = \infty$
2. $\lim\limits_{x\to c}[f(x)g(x)] = \infty, L > 0$
   $\lim\limits_{x\to c}[f(x)g(x)] = -\infty, L < 0$
3. $\lim\limits_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits.

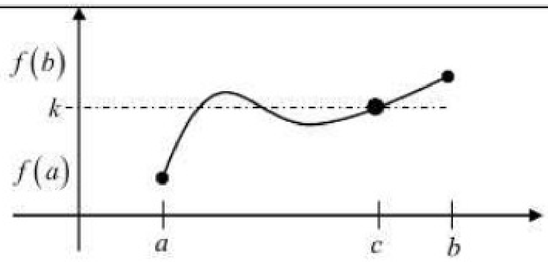
Special Limit Theorems

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ $\quad$ $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$ $\quad$ $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

You must learn (memorize) and be able to apply these.

Intermediate Value Theorem (IVT)

If f is continuous on a closed interval $[a,b]$ and $f(a) \neq f(b)$, then for every value k between $f(a)$ and $f(b)$, there is at least one value $c \in (a,b)$ such that $f(c) = k$.



- IVT roughly says that a continuous function cannot skip values.
- It also can prove that a desired value can exist between two other values, if and only if, the function is continuous.

Here's an example to make sense of this…

Consider a plane that takes off and climbs from 0 to 10,000 metres in 20 minutes. The plane must reach every altitude between 0 and 10,000 metres during this 20-minute interval. Thus, at some moment, the plane's altitude must have been exactly 8371 metres. Of course, this assumes that the plane's motion is continuous, so its altitude cannot jump abruptly from, say, 8000 to 9000 metres.

Example #6 – Verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

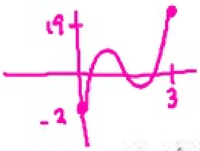
$f(x) = x^3 - x^2 + x - 2;\ [0,3] \quad f(c) = 4$

$f(x)$ is continuous on $[0,3]$ $\therefore$ the IVT applies

$f(0) = -2$
$f(3) = 19$

$\therefore$ by the IVT there exists some c-value on $[0,3]$ such that $f(c) = 4$.

To find c:
$f(c) = 4$
$f(c) - 4 = 0$
$c = 2$



Existence of Zeros

If $f(x)$ is continuous on $[a,b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then $f(x)$ has a zero in (a,b).

Example #7 – Show that $f(x) = \cos^2 x - 2\sin\frac{x}{4}$ has a zero in $[0,2]$. Then locate the zero using your calculator.

$f(x)$ is continuous on $[0,2]$ $\therefore$ the IVT applies

$f(0) = 1$
$f(2) = -0.7857$

$\therefore$ by the IVT there exist at least one zero on $[0,2]$.

$f(x) = 0$
$x = 0.8593$